

# Some Ramsey results for the $n$ -cube

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## Abstract

In this note we establish a Ramsey-type result for certain subsets of the  $n$ -dimensional cube. This can then be applied to obtain reasonable bounds on various related structures, such as (partial) Hales-Jewett lines for alphabets of sizes 3 and 4, Hilbert cubes in sets of real numbers with small sumsets, “corners” in the integer lattice in the plane, and 3-term integer geometric progressions.

## 1 Preliminaries

Let us define

$$\{0, 1\}^n = \{(\epsilon_1, \epsilon_2, \dots, \epsilon_n), \epsilon_i = 0 \text{ or } 1, 1 \leq i \leq n\}, \quad D(n) := \{0, 1\}^n \times \{0, 1\}^n.$$

We can think of the points of  $\{0, 1\}^n$  as vertices of an  $n$ -cube  $Q^n$ , and  $D(n)$  as all the line segments joining two vertices of  $Q^n$ . We will ordinarily assume that the two vertices are distinct. We can represent the points  $(X, Y) \in D(n)$  schematically by the diagram shown in Figure 1.

In the diagram, we place  $X$  to the left of  $X'$  if  $w(X) < w(X')$  where  $w(Z)$  denotes the number of 1's in the binary  $n$ -tuple  $Z$ . Similarly, we place  $Y'$  below  $Y$  if  $w(Y') < w(Y)$ . (If  $w(X) = w(X')$  or  $w(Y) = w(Y')$ , then the order doesn't matter).

With  $[n] := \{1, 2, \dots, n\}$ ,  $I \subseteq [n]$  and  $\bar{I} = [n] \setminus I$ , a **line**  $L = L(I, C)$  consists all the pairs  $((x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n))$  where  $C = (c_j)_{j \in \bar{I}}$  with  $y_i = 1 - x_i$  if  $i \in I$ , and  $x_j = y_j = c_j$  if  $j \in \bar{I}$ .

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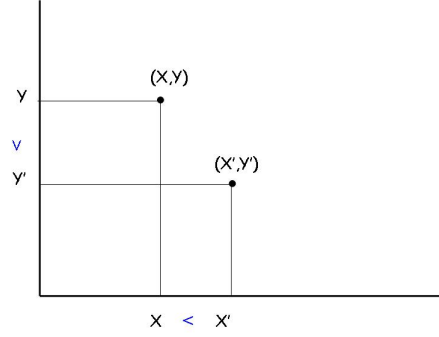


Figure 1: Representing points in  $D(n)$ .

Thus,

$$|L(I, C)| = 2^{|I|}.$$

In this case we say that  $I$  has *dimension*  $|I|$ .

**Fact.** Every point  $(X, Y) \in D(n)$  lies on a unique line.

**Proof.** Just take  $I = \{i \in [n] : x_i \neq y_i\}$  and  $c_j = x_j = y_j$  for  $j \in \bar{I}$ .

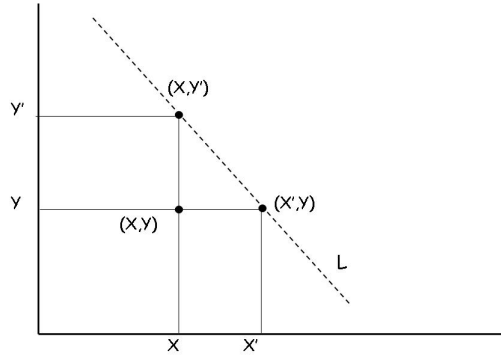


Figure 2: A corner in  $D(n)$ .

By a *corner* in  $D(n)$ , we mean a set of three points of the form  $(X, Y), (X', Y), (X, Y')$  where  $(X, Y')$  and  $(X', Y)$  are on a common line  $L$  (see Figure 2).

We can think of a corner as a binary tree with one level and root  $(X, Y)$ . More generally, a *binary tree*  $B(m)$  with  $m$  levels and root  $(X, Y)$  is defined by joining  $(X, Y)$  to the roots of two binary trees with  $m - 1$  levels. All of the  $2^k$  points at level  $k$  are required to be a common line (see Figure 3).

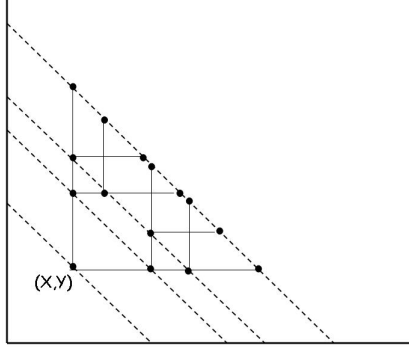


Figure 3: A binary tree with 3 levels

## 2 The main result.

Our first theorem is the following.

**Theorem 1.** For all  $r$  and  $m$ , there is an  $n_0 = n_0(r, m)$  such if  $n \geq n_0$  and the points of  $D(n)$  are arbitrarily  $r$ -colored, then there is always a monochromatic binary tree  $B(m)$  with  $m$  levels formed. In fact, we can take  $n_0(r, m) = c 6^{rm}$  for some absolute constant  $c$ .

**Proof.** Let  $n$  be large (to be specified later) and suppose the points of  $D(n)$  are  $r$ -colored. Consider the  $2^n$  points on the line  $L_0 = L([n])$ . Let  $S_0 \subseteq L_0$  be the set of points having the “most popular” color  $c_0$ . Thus,  $|S_0| \geq \frac{2^n}{r}$ . Consider the *grid*  $G_1$  (lower triangular part of a Cartesian product) defined by:

$$G_1 = \{(X, Y') : (X, Y) \in S_0, (X', Y') \in S_0 \text{ with } X < X'\}.$$

(See Figure 4).

Thus,

$$|G_1| \geq \binom{S_0}{2} > \frac{1}{4} |S_0|^2 \geq \frac{1}{4r^2} \cdot 4^n := \alpha_1 4^n.$$

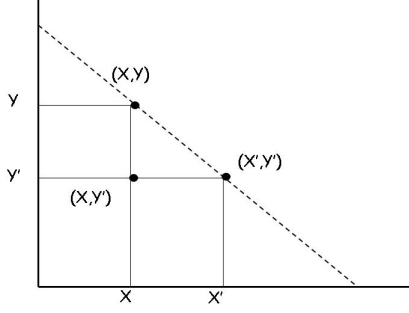


Figure 4: A grid point

Let us call a line  $L$  of dimension  $t$  *small* if  $t < \frac{n}{3}$  and *deficient* if  $|L \cap G_1| \leq (\frac{\alpha_1}{4})2^t$ .

Thus, the total number of points on small or deficient lines is at most

$$\begin{aligned}
& \sum_{t < \frac{n}{3}} 2^t \binom{n}{t} 2^{n-t} + \sum_{t \geq \frac{n}{3}} \frac{\alpha_1}{4} 2^t \binom{n}{t} 2^{n-t} \\
&= \frac{\alpha_1}{4} \sum_t 2^n \binom{n}{t} + (1 - \frac{\alpha_1}{4}) \sum_{t < \frac{n}{3}} 2^n \binom{n}{t} \\
&\leq (\frac{\alpha_1}{4})4^n + (1 - \frac{\alpha_1}{4})(3.8^n) \\
&\leq (\frac{\alpha_1}{2})4^n \quad (\text{since } \sum_{t < \frac{n}{3}} \binom{n}{t} < 1.9^n \text{ follows easily by induction})
\end{aligned}$$

provided  $\alpha_1 \geq 2 \cdot (.95)^n$ .

Thus, if we discard these points, we still have at least  $(\frac{\alpha_1}{2})4^n$  points remaining in  $G_1$ , and all these points are on “good” lines, i.e., not small and not deficient. Let  $L_1$  be such a good line, say of dimension  $|L_1| = n_1 \geq \frac{n}{3}$ . Let  $S_1$  denote the set of points of  $L_1 \cap G_1$  with the most popular color  $c_1$ . Therefore

$$|S_1| \geq (\frac{\alpha_1}{4r})2^{n_1}.$$

Observe that  $G_2 \subset G_1$  (see Figure 5).

Now let  $G_2$  denote the “grid” formed by  $S_1$ , i.e.,

$$G_2 = \{(X, Y') : (X, Y \in S_1, (X', Y') \in S_1, \text{ with } X < X')\}.$$

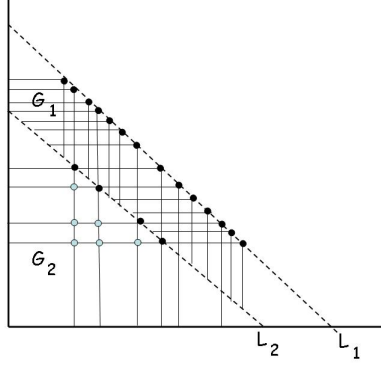


Figure 5:  $G_2 \subset G_1$

Therefore, we have

$$|G_2| \geq \binom{|S_1|}{2} \geq \left(\frac{\alpha_1}{8r}\right)^2 4^{n_1} := \alpha_2 4^{n_1}.$$

As before, let us classify a line  $L$  of dimension  $t$  as *small* if  $t < \frac{n_1}{3}$ , and as *deficient* if  $|L \cap G_2| \leq \left(\frac{\alpha_2}{4}\right) 2^t$ .

A similar calculation as before shows that if we remove from  $G_2$  all the points on small or deficient lines, then at least  $\left(\frac{\alpha_2}{2}\right) 4^{n_1}$  points will remain in  $G_2$ , provided  $\alpha_2 \geq 2 \cdot (.95)^{n_1}$ .

Let  $S_2 \subseteq L_2 \cap G_2$  have the most popular color  $c_2$ , so that

$$|S_2| \geq \left(\frac{\alpha_2}{4r}\right) 2^{n_2}.$$

Then, with  $G_3$  defined to be the “grid” formed by  $S_2$ , we have  $|G_3| \geq \left(\frac{\alpha_2}{8r}\right)^2 4^{n_2}$ , and so on. Note that  $G_3 \subset G_2 \subset G_1$ .

We continue this process for  $rm$  steps.

In general, we define

$$\alpha_{i+1} = \left(\frac{\alpha_i}{8r}\right)^2, i \leq rm - 1$$

with  $\alpha_1 = \frac{1}{4r^2}$ . By construction, we have  $n_{i+1} \geq \frac{n_i}{3}$  for all  $i$ . In addition, we will need to have  $\alpha_i \geq 2 \cdot (.95)^{n_i}$  for all  $i$  for the argument to be valid. In particular, this implies that in general

$$\alpha_k = \frac{1}{2^{2^{k+2}-6} r^{2^{k+1}-2}}.$$

It is now straightforward to check that all the required inequalities are satisfied by choosing  $n \geq n_0(r, m) = c \cdot 6^{rm}$  for a suitable absolute constant  $c$ .

Hence, there must be  $m$  indices  $i_1 < i_2 < \dots < i_m$  such that all the sets  $S_{i_k}$  have the same color.

These  $m$  sets  $S_{i_k}$  contain the desired monochromatic binary tree  $B(m)$ .

### 3 Some interpretations

#### 3.1 Self-crossing paths

As we stated at the beginning, we can think of  $D(n)$  as the set of all the diagonals of the  $n$ -cube  $Q^n$ . Let us call a pair  $\{x, \bar{x}\} = \{(x_1, \dots, x_n), (\bar{x}_1, \dots, \bar{x}_n)\}$  a **main** diagonal of  $Q^n$  where  $\bar{x}_i = 1 - x_i$ .

An affine  $k$ -subcube of  $Q^n$  is defined to be a subset of  $2^k$  points of the form  $\{(y_1, \dots, y_n) : y_i = 0 \text{ or } 1 \text{ if and only if } i \in I\}$  for some  $k$ -subset  $I \subseteq [n] = \{1, 2, \dots, n\}$ .

We will say that three connected diagonals of the form  $\{x, y\}, \{y, z\}, \{z, w\}$  form a *self-crossing path*, denoted by  $\bowtie$ , if  $\{x, y\}$  and  $\{z, w\}$  are both main diagonals of the same subcube.

**Corollary 1.** In any  $r$ -coloring of the edges in  $D(n)$ , there is always a monochromatic self-crossing path  $\bowtie$ , provided  $n > c \cdot 6^r$  (where  $c$  is a suitable absolute constant).

The same argument works for any subgraph  $G$  of  $D(n)$ , provided that  $G$  has enough edges and for any pair of crossing main diagonals,  $G$  has all the edges between the pair's endpoints.

#### 3.2 Corners

The preceding techniques can be used to prove the following.

**Theorem 3.** For every  $r$ , there exists  $\delta = \delta(r)$  and  $n_0 = n_0(r)$  with the following property:

If  $A$  and  $B$  are sets of real numbers with  $|A| = |B| = n \geq n_0$  and  $|A+B| \leq n^{1+\delta}$ , then any  $r$ -coloring of  $A \times B$  contains a monochromatic “corner”, i.e., a set of 3 points of the form  $(a, b), (a + d, b), (a, b + d)$  for some positive number  $d$ . In fact, the argument shows that we can choose  $\delta = \frac{1}{2^{r+1}}$ .

The calculation goes as follows; The Cartesian product  $A \times B$  can be covered

by  $n^{1+\delta}$  lines of slope -1. Choose the line with the most points from  $A \times B$ , denoted by  $L_0$ . There are at least  $n^2/n^{1+\delta}$  points in  $L_0 \cap A \times B$ . Choose the set of points  $S_1$  with the most popular color in  $L_0 \cap A \times B$ . ( $|S_1| \geq n^{1-\delta}/r$ ) As before, consider the grid  $G_1$  defined by  $S_1$ , and choose the slope -1 line,  $L_2$ , which has the largest intersection with  $G_1$ . Choose the set of points,  $S_2$ , having the most popular color and repeat the process with  $G_2$ , the grid defined by  $S_2$ . We can't have more than  $r$  iterations without having a monochromatic corner. Solving the simple recurrence  $a_{n+1} = 2a_n + 1$  in the exponent, one can see that after  $r$  steps the size of  $S_r$  is at least  $c_r n^{1-\delta(2^{r+1}-1)}$ . If this quantity is at least 2, then we have at least one more step and the monochromatic corner is unavoidable. The inequality

$$c_r n^{1-\delta(2^{r+1}-1)} \geq 2$$

can be rearranged into

$$n^{1-\delta 2^{r+1}} \geq \frac{2}{c_r n^\delta}.$$

From this we see that choosing  $\delta = 2^{-r-1}$  guarantees that for large enough  $n$  the inequality above is true, proving our statement.

By iterating these techniques, one can show that the same hypotheses on  $|A|$  and  $|B|$  (with appropriate  $\delta = \delta(r, m)$  and  $n_0 = n_0(r, m)$ ), imply that if  $A \times B$  is  $r$ -colored then each set contains a monochromatic translate of a large ‘‘Hilbert cube’’, i.e., a set of the form

$$H_m(a, a_1, \dots, a_m) = \{a + \sum_{1 \leq i \leq m} \epsilon_i a_i\} \subset A,$$

$$H_m(b, a_1, \dots, a_m) = \{b + \sum_{1 \leq i \leq m} \epsilon_i a_i\} \subset B$$

where  $\epsilon_i = 0$  or  $1$ ,  $1 \leq i \leq m$ .

### 3.3 Partial Hales-Jewett lines

**Corollary 2.** For every  $r$  there is an  $n = n_0(r) \leq c6^r$ , with the following property. For every  $r$ -coloring of  $\{0, 1, 2, 3\}^n$  with  $n > n_0$ , there is always a monochromatic set of 3 points of the form:

$$\begin{aligned} &(\dots, a, \dots, 0, \dots, b, \dots, 3, \dots, 0, \dots, c, \dots, 3, \dots, d, \dots) \\ &(\dots, a, \dots, 1, \dots, b, \dots, 2, \dots, 1, \dots, c, \dots, 2, \dots, d, \dots) \\ &(\dots, a, \dots, 2, \dots, b, \dots, 1, \dots, 2, \dots, c, \dots, 1, \dots, d, \dots) \end{aligned}$$

In other words, every column is either *constant*, *increasing* from 0, or *decreasing* from 3.

**Proof.** To each point  $(x_1, x_2, \dots, x_n)$  in  $\{0, 1, 2, 3\}^n$ , we associate the point  $((a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n))$  in  $\{0, 1\}^n \times \{0, 1\}^n$  by the following rule:

$x_k$	$\leftrightarrow$	$a_k$	$b_k$
0		0	0
1		0	1
2		1	0
3		1	1

Then it not hard to verify that a monochromatic corner in  $D(n)$  corresponds to a monochromatic set of 3 points as described above, a structure which we might call a partial Hales-Jewett line.

**Corollary 3.** For every  $r$  there is an  $n = n_0(r) \leq c6^r$ , with the following property. For every  $r$ -coloring of  $\{0, 1, 2\}^n$  with  $n > n_0$ , there is always a monochromatic set of 3 points of the form:

$$\begin{aligned} &(\dots, a, \dots, 0, \dots, b, \dots, 0, \dots, 0, \dots, c, \dots, 0, \dots, d, \dots) \\ &(\dots, a, \dots, 1, \dots, b, \dots, 2, \dots, 1, \dots, c, \dots, 2, \dots, d, \dots) \\ &(\dots, a, \dots, 2, \dots, b, \dots, 1, \dots, 2, \dots, c, \dots, 1, \dots, d, \dots) \end{aligned}$$

**Proof.** Map the points  $(a_1, a_2, \dots, a_n) \in \{0, 1, 2, 3\}$  to points  $(b_1, b_2, \dots, b_n) \in \{0, 1, 2\}^n$  by:

$$a_i = 0 \text{ or } 3 \Rightarrow b_i = 0, a_i = 1 \Rightarrow b_i = 1, a_i = 2 \Rightarrow b_i = 2$$

The theorem now follows by applying Corollary 2. □

### 3.4 3-term geometric progressions.

The simplest non-trivial case of van der Waerden's theorem [6] states that for any natural number  $r$ , there is a number  $W(r)$  such that for any  $r$ -coloring of the first  $W(r)$  natural numbers there is a monochromatic three-term arithmetic progression. Finding the exact value of  $W(r)$  for large  $r$ -s is a hopelessly difficult task. The best upper bound follows from a recent result of Bourgain [1];

$$W(r) \leq ce^{r^{3/2}}.$$

One can ask the similar problem for geometric progressions; What is the maximum number of colors, denoted by  $r(N)$ , that for any  $r(N)$ -coloring of the first  $N$  natural numbers there is a monochromatic geometric progression. Applying Bourgain's bound to the exponents of the geometric progression  $\{2^i\}_{i=0}^\infty$ , shows that  $r(N) \geq c \log \log N$ . Using our method we can obtain the same bound, without applying Bourgain's deep result.



Observe that if we associate the point  $(a_1, a_2, \dots, a_k, \dots, a_n)$  with the integer  $\prod_k p_k^{a_k}$ , where  $p_i$  denotes the  $i^{th}$  prime, then the points

$$\begin{aligned} &(\dots, a, \dots, 0, \dots, b, \dots, 3, \dots, 0, \dots, c, \dots, 3, \dots, d, \dots) \\ &(\dots, a, \dots, 1, \dots, b, \dots, 2, \dots, 1, \dots, c, \dots, 2, \dots, d, \dots) \\ &(\dots, a, \dots, 2, \dots, b, \dots, 1, \dots, 2, \dots, c, \dots, 1, \dots, d, \dots) \end{aligned}$$

correspond to a 3-term geometric progression. Our bound from Corollary 2 with an estimate for the product of the first  $n$  primes imply that  $r(N) \geq c \log \log N$ .

## 4 Concluding remarks

It would be interesting to know if we can “complete the square” for some of these results. For example, one can use these methods to show that if the points of  $[N] \times [N]$  are colored with at most  $c \log \log N$  colors, then there is always a monochromatic “corner” formed, i.e., 3 points  $(a, b)$ ,  $(a', b)$ ,  $(a, b')$  with  $a' + b = a + b'$ . By projection, this gives a 3-term arithmetic progression (see [3]).

Is it the case that with these bounds (or even better ones), we can guarantee the  $4^{th}$  point  $(a', b')$  to be monochromatic as well?

Similarly, if the diagonals of an  $n$ -cube are  $r$ -colored with  $r < c \log \log n$ , is it true that a monochromatic  $\bowtie$  must be formed, i.e., a self-crossing 4-cycle (which is a self-crossing path with one more edge added)?

Let  $\boxtimes$  denote the structure consisting of the set of 6 edges spanned by 4 **coplanar** vertices of an  $n$ -cube. In this case, the occurrence of a monochromatic  $\boxtimes$  is guaranteed once  $n \geq N_0$ , where  $N_0$  is a **very** large (but well defined) integer, sometimes referred to as Graham’s number (see [5]). The best lower bound currently available for  $N_0$  is 11 (due to G. Exoo [2]).

One can also ask for estimates for the density analogs for the preceding results. For example, Shkredov has shown the following:

**Theorem** ([4]). Let  $\delta > 0$  and  $N \ll \exp \exp(\delta^{-c})$ , where  $c > 0$  is an absolute constant. Let  $A$  be a subset of  $\{1, 2, \dots, N\}^2$  of cardinality at least  $\delta N^2$ . Then  $A$  contains a corner.

It would be interesting to know if the same hypothesis implies that  $A$  contains the  $4^{th}$  point of the corner, for example.

## References

- [1] J. Bourgain, *Roth’s theorem on progressions revisited*, Journal d’Analyse Mathématique 104(1) (2008), 155–192.

- [2] G. Exoo (personal communication).
- [3] R. Graham and J. Solymosi, *Monochromatic right triangles on the integer grid*, Topics in Discrete Mathematics, Algorithms and Combinatorics **26** (2006), 129–132.
- [4] I. D. Shkredov, *On a Generalization of Szemerédi’s Theorem*, Proceedings of the London Mathematical Society 2006 93(3):723–760.
- [5] [http://en.wikipedia.org/wiki/Grahams\\_Number](http://en.wikipedia.org/wiki/Grahams_Number).
- [6] B.L. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw. Arch. Wisk. 15 (1927), 212–216.